Pergamon

Appl. Math. Lett. Vol. 7, No. 6, pp. 85–89, 1994

0893-9659/94 $7.00 + 0.00
0893-9659(94)00097-2

Asymptotic Behaviour in Time of KdV Type Equations with Time Dependent Coefficients

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(Received and accepted September 1993)

Abstract—We study the asymptotic behaviour in time of the solutions of a class of evolution equations whose simplest representative would be the Korteweg de Vries equation with variable coefficients. Specific rates of decay are given in either the "conservative" or the dissipative case.

Keywords—Decay rates, Pseudo-differential operators, Energy method.

1. INTRODUCTION

In this paper, we shall study the asymptotic behaviour in time of the solutions of a family of evolution equations whose simplest representative would be the well known Korteweg de Vries equation with variable coefficients. The model arises in a natural way while considering long waves which take place on a free-surface of a horizontal layer of fluid with finite depth subject to the (possible) effect of unevenness of the bottom surface. An additional dissipative term may or may not be present in the model (see [1,2]). The above motivation leads us to consider the Cauchy problem:

$$u_t - a(t)Mu_x + b(t)u^p u_x + \alpha Lu = 0, \qquad u(x,0) = \varphi(x), \tag{1.1}$$

where $-\infty < x < +\infty$, $t \geq 0$ and M and L are defined as Fourier multiplier operators by

$$\widehat{Mf}(y) = m(y)\,\widehat{f}(y), \qquad \text{whenever } f \in H^\mu(\mathbb{R}) \tag{1.2}$$

$$\widehat{Lg}(y) = \ell(y)\,\widehat{g}(y), \qquad \text{whenever } g \in H^s(\mathbb{R}) \tag{1.3}$$

for all $y \in \mathbb{R}$. Here and in the sequel, circumflexes will be use to indicate Fourier transform with respect to the spatial variable x. As usual, $H^r(\mathbb{R})$ denotes the Sobolev space of order r. In (1.1) $a(t)$ and $b(t)$ are real-valued and continuous functions of t, p is an integer (varying in a suitable interval) and $\alpha \geq 0$. The existence of global solutions of (1.1) has been intensively studied by several authors in recent years (see [3–5] and the references therein). Actually, in [3] more general nonlinearities were considered.

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Concerning the asymptotic behaviour in time, P. Biler [6] considered problem (1.1) with constant coefficients. In the dissipative case (i.e., $\alpha > 0$) our final conclusions are essentially the same as his, but in the "conservative" case (i.e., $\alpha = 0$), our result takes into consideration the effect of unevenness of the bottom surface, the order of the symbols and their relation with p. Roughly speaking, we prove in Section 3 that, under suitable assumptions on $a(t)$ and $b(t)$, the solution u of (1.1) with $\alpha = 0$, satisfies $|u(\cdot,t)|_\infty \le C(1+A(t))^{-(1+2\mu)^{-1}}$ as $t \to +\infty$, provided the initial data is sufficiently small. Here, $A(t) = \int_0^t a(\tau)\,d\tau$ and 2μ is the order of the symbol $m(y)$. We shall use standard notation: We denote the $L^r(\mathbb{R})$ norm by $|\cdot|_r$, $1 \le r \le +\infty$. The Sobolev space $H^j(\mathbb{R})$ norm by $\|\ \|_j$, $0 \le j < +\infty$. Various positive constants will be denoted by C but, we remark that they may vary from line to line. We shall assume that $a(t)$, $b(t)$, $m(y)$, $\ell(y)$, p, and φ satisfy all the appropriate conditions to have a global solution $u = u(x,t)$ of (1.1). Thus, $m(y)$ and $\ell(y)$ are positive and satisfy suitable growth conditions (see [5,7]) of the type $0 < c_1(1+|y|)^\mu \le m(y) \le c_2(1+|y|)^\mu$ and $0 \le c_3|y|^s \le \ell(y) \le c_4|y|^s$ for some positive constants c_j, $j = 1,2,3,4$ and all $y \in \mathbb{R}$.

2. THE DISSIPATIVE CASE ($\alpha > 0$)

LEMMA 1. *Let u be the global solution of problem (1.1) with $\alpha > 0$ and $\varphi \in H^j(\mathbb{R})$, ($j \ge 2$), $2s > p+2$, $s \ge 2$. Furthermore, suppose that m and ℓ are even functions, $a(t)$, $b(t)$ are real-valued and continuous functions with $b \in L^\infty(\mathbb{R})$ then*

(1) $|L^{1/2}u(\cdot,t)|_2 \to 0$

(2) $|u(\cdot,t)|_\infty \to 0$ *as* $t \to +\infty$.

PROOF. Multiply equation (1.1) by u, integrate in space to obtain

$$\frac{d}{dt}\,|u(\cdot,t)|_2^2 + 2\alpha\,|L^{1/2}\,u(\cdot,t)|_2^2 = 0 \tag{2.1}$$

because $\int_{-\infty}^{\infty} uMu_x d_x = 0$ since $m(y)$ is even. Integration in time of (2.1) gives us

$$|u(\cdot,t)|_2^2 + 2\alpha\int_0^t |L^{1/2}u(\cdot,\tau)|_2^2\,d\tau = |\varphi|_2^2 \tag{2.2}$$

for any $t \ge 0$. Next, we multiply (1.1) by Lu and integrate in space to obtain

$$\frac{d}{dt}\,|L^{1/2}u(\cdot,t)|_2^2 + 2\alpha|Lu(\cdot,t)|_2^2 = -2b(t)\int_{-\infty}^{\infty} u^p u_x Lu\,dx \tag{2.3}$$

because $\int_{-\infty}^{\infty} LuMu_x\,dx = 0$ since $\ell(y)$ and $m(y)$ are even. Next, we estimate the right hand side of (2.3). Using Schwarz's inequality and $2AB \le A^2 + B^2$, we obtain

$$-2b(t)\int_{-\infty}^{\infty} u^p u_x Lu\,dx \le 2|b(t)|\,|Lu|_2|u^p u_x|_2 \le \alpha|Lu|_2^2 + \frac{1}{\alpha}\,|b(t)|^2|u^p u_x|_2^2 \tag{2.4}$$

From (2.3), (2.4) and the imbedding $H^j \hookrightarrow L^\infty$ it follows that

$$\frac{d}{dt}|L^{1/2}u(\cdot,t)|_2^2 + \alpha|Lu(\cdot,t)|_2^2 \le C|b(t)|^2|u|_\infty^{2p}|u_x|_2^2 \le C|b(t)|^2|\varphi|_2^{p+\frac{(p+2)(s-1)}{s}}\ |Lu|^{(p+2)/s} \tag{2.5}$$

where we use Gagliardo-Niremberg's inequality. Using $AB \le (A^{p_1})/p_1 + (B^{q_1})/q_1$ with $q_1 = (2s)/(p+2)$, $p_1 = (2s)/(2s-p-2)$, $2s > p+2$, $B = |Lu|_2$, it follows from (2.5) that $\frac{d}{dt}|L^{1/2}u(\cdot,t)|_2^2 \le C$ which, together with (2.2) implies item (1) of the lemma. Using Gagliardo-Niremberg's inequality and (2.2), we deduce that

$$|u|_\infty \le c|Lu|_2^{1/s}|u|_2^{1-(1/s)} \le C|Lu|_2^{1/s}|\varphi|_2^{1-(1/s)} \to 0, \qquad \text{as } t \to +\infty$$

which proves item (2) of the lemma.

LEMMA 2. *Let $G = G(x,t)$ be the fundamental solution of the linear problem $u_t - a(t)Mu_x + \alpha Lu = 0$, with $-\infty < x < \infty$, $t > 0$, $\alpha > 0$ and a, b, M and L as in Lemma 1, then*

(1) $t^{1/s}|G(\cdot,t)|_\infty \leq C$ *and*
(2) $t^{1/(2s)}|G(\cdot,t)|_2 \leq C$.

PROOF. Via Fourier transform, G can be written as

$$G(x,t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \exp\left(ixy - \alpha\ell(y)t - iym(y)A(t)\right)\, dy, \tag{2.6}$$

where $A(t) = \int_0^t a(\tau)\, d\tau$. From (2.6), the behaviour of $\ell(y)$ and the change of variable $z = \alpha t C_3 y^s$, we obtain

$$t^{1/s}|G(x,t)| \leq Ct^{1/s}\int_{-\infty}^{\infty} e^{-\alpha\ell(y)t}dy \leq Ct^{1/s}\int_0^{\infty} e^{-\alpha C_3 t y^s}dy = C\int_0^{\infty} e^{-z}z^{(1/s)-1}dz = C\Gamma\left(\frac{1}{s}\right)$$

which proves item (1). Item (2) follows in similar manner.

THEOREM 1. *Let u be the global solution of* (1.1). *Under the assumptions of Lemma 2, there exist a positive constant C such that*

$$|u(\cdot,t)|_\infty \leq C(1+t)^{-1/2s}, \qquad \text{for large } t.$$

PROOF. Fix $T > 0$ large. For $t \geq T$, we use Duhamel's theorem to write u as

$$u(x,t) = G(x,t-T) * u(x,T) - \int_T^t G(\cdot, T-\tau) * b(\tau)u^p u_x\, d\tau. \tag{2.7}$$

Taking L^∞ norm, using Lemma 2 and convolution inequalities in (2.7) it follows that

$$|u(\cdot,t)|_\infty \leq C(t-T)^{-1/2s}|u(\cdot,T)|_2 + C\int_T^t (t-\tau)^{-1/s}|b(\tau)|\, |u^p u_x|_1\, d\tau. \tag{2.8}$$

Using the imbedding $H^j \hookrightarrow L^\infty$, (2.2), and Gagliardo-Niremberg's inequality, we can estimate $|u^p u_x|_1$ as follows

$$|u^p u_x|_1 \leq |u|_\infty^{p-1}|u|_2|u_x|_2 \leq C(\varphi)|u|_\infty^{p-1}|L^{1/2}u|_2^{1/s}.$$

Consequently, from (2.8), we obtain that

$$|u(\cdot,t)|_\infty \leq C(t-T)^{-1/2s}|u(\cdot,T)|_2 + C\varepsilon\int_T^t (t-\tau)^{-1/s}|b(\tau)|\, |u|_\infty^{p-1}\, d\tau, \tag{2.9}$$

where we used Lemma 1 choosing T large enough so that $|L^{1/2}u|_2^{1/s} \leq \varepsilon$ for $t \geq T$. Let $g(t) = \sup_{T\leq\sigma\leq t}(1+\sigma-T)^{1/(2s)}|u(\cdot,\sigma)|_\infty$, then it follows from (2.9) that $g(t) \leq C|u(\cdot,T)|_2 + C\varepsilon|b|_\infty g^{p-1}(t)$. Since $\varepsilon > 0$ is small, then g is bounded. In particular, $(1+t)^{1/(2s)}|u(\cdot,t)|_\infty$ is bounded for large t.

3. THE CASE $\alpha = 0$

In this section, we consider the Cauchy problem

$$u_t - a(t)Mu_x + b(t)u^p u_x = 0, \qquad u(x,0) = \varphi(x), \tag{3.1}$$

where $-\infty < x < \infty$ and $t \geq 0$, $\varphi \in W^{1,1}(\mathbb{R}) \cap H^2(\mathbb{R})$ and $a(t)$ and $b(t)$ are continuous functions and $a(t) > 0$. First, we consider the linear problem

$$v_t - a(t)Mv_x = 0, \qquad v(x,0) = \varphi(x). \tag{3.2}$$

Estimates of the fundamental solution of (3.2) are in general, quite complicated to obtain, therefore, we restrict ourselves to the case where $M = -\left(\frac{\partial^2}{\partial x^2}\right)^\mu$ with $\mu \geq 1$. The discussion given in [4] in case $a \equiv 1$ could be extended provided that $\int_0^\infty a(\sigma)\,d\sigma = +\infty$ to show that

$$|v(\cdot,t)|_\infty \leq C|\varphi|_1(A(t))^{-(1+2\mu)^{-1}} \tag{3.3}$$

for large t, where $A(t) = \int_0^t a(\tau)\,d\tau$. Since v satisfies the estimate $|v(\cdot,t)|_\infty \leq C\|\varphi\|_2$, we can write

$$|v(\cdot,t)|_\infty \leq C\left(|\varphi|_1 + \|\varphi\|_2\right)\,(1+A(t))^{-(1+2\mu)^{-1}}, \qquad \text{for large } t.$$

Let us observe that if $a(t) > 0$ and $\omega(x,t)$ is the solution of

$$\omega_t - M\omega_x + \frac{b(t)}{a(t)}\omega^p\omega_x = 0, \qquad \omega(x,0) = \varphi(x), \tag{3.4}$$

then $u(x,t) = \omega(x, A(t))$ satisfies (3.1) where $A(t) = \int_0^t a(\tau)\,d\tau$.

The above observation help us prove the intermediate step: If the initial data φ in problem (3.1) is sufficiently small, say, in the norm of $W^{1,1} \cap H^2$, and the coefficients satisfy suitable assumptions, then $|u_x(\cdot,t)|_2$ as well as $|u(\cdot,t)|_\infty$ are arbitrary small. Sufficient conditions for this to hold are $b/a \in L^\infty(\mathbb{R}^+)$ and $(b/a)^{(1)} \in L^1(\mathbb{R}^+)$. To prove this fact, we perform estimates for the solution of (3.4) borrowing ideas of the papers of J-C. Saut [3] and W. Strauss [8].

Using the results of C. Kenig *et al.* [4], our final conclusion is:

THEOREM 2. *Let* $p > \max\{2\mu+2, \mu+(7/2)\} - 1$ *with* $\mu \geq 1$, *then there exist* $\delta > 0$ *such that if* $\varphi \in W^{1,1}(\mathbb{R}) \cap H^2(\mathbb{R})$ *and* $\|\varphi\|_{W^{1,1}} + \|\varphi\|_2 < \delta$ *and* M *as above,* a *and* $b \in C(\mathbb{R}^+)$, $a > 0$ *and* b, $1/a \in L^\infty(\mathbb{R}^+)$, $(b/a)^{(1)} \in L^1(\mathbb{R}^+)$ *and* $\int_0^\infty a\,d\sigma = +\infty$, *then the solution of* (3.1) *satisfies*

$$|u(\cdot,t)|_\infty \leq C(1+A(t))^{-(1+2\mu)^{-1}}$$

for some positive constant C *and large* t. *Here* $A(t) = \int_0^t a(\tau)\,d\tau$.

PROOF. The solution u of (3.1) can be written as

$$u = v - \int_0^t G(\cdot, t-\tau) * b(\tau)u^p u_x\,d\tau, \tag{3.5}$$

where v solves problem (3.2). Using (3.3) and convolution inequalities, we obtain from (3.5) as we did during the proof of Theorem 1:

$$\begin{aligned}|u(\cdot,t)|_\infty &\leq C(|\varphi|_1 + \|\varphi\|_2)(1+A(t))^{-(1+2\mu)^{-1}} \\ &\quad + C\int_0^t (1+A(t)-A(\tau))^{-(1+2\mu)^{-1}}|b(\tau)|\,|u|_\infty^{p-1}|u_x|_2\,d\tau.\end{aligned}$$

Let $g(t) = \sup_{0\leq\sigma\leq t}(1+A(\sigma))^{(1+2\mu)^{-1}}|u(\cdot,\sigma)|_\infty$, then, using the above observation and (3.6), we deduce that the integrand in (3.6) is least or equal to

$$C(\varphi)g^{p-1}(t)|b|_\infty(1+A(t)-A(\tau))^{-(1+2\mu)^{-1}}(1+A(\tau))^{-(1+2\mu)^{-1}(p-1)},$$

which implies that $g(t) \leq C(|\varphi|_1 + \|\varphi\|_2) + C(\varphi)g^{p-1}(t)$. Since $C(\varphi)$ is small, we conclude that $g(t)$ is bounded, therefore, the conclusion of Theorem 2 holds.

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